BIFURCATION OF EQUILIBRIUM OF A PERFECTLY ELASTIC BODY (*)

(BIFURKATSIIA RAVNOVESIIA IDEAL'NO UPRUGOGO TELA)

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Kirchhoff's well known theorem on the uniqueness of the equilibrium state of a linearly elastic medium makes it necessary to regard any attempt at constructing a theory of bifurcation of equilibrium for such a medium as logically unjustifiable. Construction of a theory of bifurcation must necessarily be based on the investigation of equilibrium states which are close to the equilibrium state being studied for a nonlinearly elastic solid.

In Section 1 a summary of notation is given and the formulas of tensor analysis which will be applied later are recalled. In Sections 2 to 4 the required geometric and statical relations are given for a nonlinearly elastic isotropic body. In Sections 5 to 7 a derivation of the equations for equilibrium states near a previously known equilibrium state is presented. This theory is applied to the case of a compressed bar considered as a three-dimensional body in Sections 8 and 9, with the simplest specification of the strain energy density expression. The bifurcation value of the parameter (the compressive force) turns out, of course, to be very near the Euler critical value, and coincides with it if the cross-sectional dimensions of the bar are neglected in comparison to its length.

1. Notation. Two states of a volume of a continuous medium are considered: an initial state (the volume v bounded by the surface s) and a final state (the volume V bounded by the surface S). A point of the medium is specified by the material coordinates q^1, q^2, q^3. Its radius vector in a fixed Cartesian system of axes $OXYZ$ which is equal to $\mathbf{r}(q^1, q^2, q^3)$ in the v-volume is transformed into the radius vector $\mathbf{R}(q^1, q^2, q^3)$ in the V-volume. The coordinate vector bases

$$\mathbf{r}_s = \frac{\partial \mathbf{r}}{\partial q^s}, \qquad \mathbf{R}_s = \frac{\partial \mathbf{R}}{\partial q^s} \tag{1.1}$$

are introduced in the v- and V-volumes, respectively. In terms of these matrices $\|g_{sk}\| = \|\mathbf{r}_s \cdot \mathbf{r}_k\|$ and $\|G_{sk}\| = \|\mathbf{R}_s \cdot \mathbf{R}_k\|$ of the covariant components

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of the metric tensors g and G of these volumes, are defined. The elements g^{sk} and G^{sk} of the inverse matrices give the contravariant components of these tensors, in terms of which the associated coordinate bases are defined

$$\mathbf{r}^s = g^{sk}\mathbf{r}_k, \qquad \mathbf{R}^s = G^{sk}\mathbf{R}_k \tag{1.2}$$

The mixed components of the metric tensors g and G are denoted, as always, by

$$g_k^{\ s} = \mathbf{r}^s \cdot \mathbf{r}_k = g^{sr}g_{rk} = \begin{cases} 1 & s = k \\ 0 & s \neq k \end{cases}$$

$$G_k^{\ s} = \mathbf{R}^s \cdot \mathbf{R}_k = G^{sr}G_{rk} = \begin{cases} 1 & s = k \\ 0 & s \neq k \end{cases}$$

and the dyadic representations of these tensors are written in the form

$$\mathbf{g} = g_{sk}\mathbf{r}^s\mathbf{r}^k = g^{sk}\mathbf{r}_s\mathbf{r}_k = \mathbf{r}^s\mathbf{r}_s = \mathbf{r}_s\mathbf{r}^s \tag{1.3}$$

$$\mathbf{G} = G_{sk}\mathbf{R}^s\mathbf{R}^k = G^{sk}\mathbf{R}_s\mathbf{R}_k = \mathbf{R}^s\mathbf{R}_s = \mathbf{R}_s\mathbf{R}^s \tag{1.4}$$

The tensor g plays the role of the unit tensor in the v-volume, as does G in the V-volume. Multiplication by g or G on the left or right of any tensor specified in the corresponding volume results in the same tensor.

Differential operations on tensors in the v- and V-volumes are carried out with the aid of the nabla operators, the symbolic vectors

$$\nabla = \mathbf{r}^s \frac{\partial}{\partial q^s}, \qquad \nabla^\circ = \mathbf{R}^s \frac{\partial}{\partial q^s}$$

For instance, the tensor of the second order which is the gradient of a vector $\mathbf{a}$ is represented by the dyadic product

$$\nabla\mathbf{a} = \mathbf{r}^s \frac{\partial \mathbf{a}}{\partial q^s} = \mathbf{r}^s\mathbf{r}^k\nabla_s a_k = \mathbf{r}^s\mathbf{r}_k\nabla_s a^k, \qquad (a_k = \mathbf{a}\cdot\mathbf{r}_k,\ a^k = \mathbf{a}\cdot\mathbf{r}^k) \tag{1.5}$$

$$\nabla\mathbf{a}^\circ = \mathbf{R}^s \frac{\partial \mathbf{a}}{\partial q^s} = \mathbf{R}^s\mathbf{R}^k\nabla_s^\circ a_k^\circ = \mathbf{R}^s\mathbf{R}_k\nabla_s^\circ a^{\circ k}, \qquad (a_k^\circ = \mathbf{a}\cdot\mathbf{R}_k,\ a^{\circ k} = \mathbf{a}\cdot\mathbf{R}^k) \tag{1.6}$$

Operations in the V-volume (with the metric tensor G) are indicated by the superscript $^\circ$ here and further on in Sections 2 to 4. Covariant differential operators are denoted by ∇_s and ∇_s° , so that

$$\nabla_s a_k = \frac{\partial a_k}{\partial q^s} - \left\{ {r \atop sk} \right\} a_r, \qquad \nabla_s^\circ a_k^\circ = \frac{\partial a_k^\circ}{\partial q^s} - \left\{ {r \atop sk} \right\}^\circ a_r^\circ$$

From what has been said we have also

$$\nabla\mathbf{a} = \mathbf{r}^s\mathbf{r}^t\nabla_s a_t = \mathbf{r}^s\mathbf{r}_t\nabla_s a^t, \qquad \nabla^\circ\mathbf{a} = \mathbf{R}^s\mathbf{R}^t\nabla_s^\circ a_t^\circ = \mathbf{R}^s\mathbf{R}_t\nabla_s^\circ a^{\circ t}$$

In addition we give the expression for the divergence of a second-order tensor Q . It is written in, say, the metric of the v-volume in the form

$$\operatorname{div} Q = \nabla\cdot Q = \mathbf{r}^s \cdot \frac{\partial}{\partial q^s}\mathbf{r}_m\mathbf{r}_t q^{mt} = \mathbf{r}_t\nabla_s q^{st} = \mathbf{r}_t\left(\frac{\partial q^{st}}{\partial q^s} + \left\{ {s \atop sr} \right\} q^{rt} + \left\{ {t \atop sr} \right\} q^{sr}\right)$$

Recalling also that

$$\left\{ {s \atop sr} \right\} = \frac{\partial\sqrt{g}}{\sqrt{g}\,\partial q^r}, \qquad \mathbf{r}_t\left\{ {t \atop sr} \right\} = \frac{\partial \mathbf{r}_r}{\partial q^s}$$

we arrive at a relation which will frequently be applied in what follows

$$\operatorname{div} Q = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^s} \sqrt{g} q^{st} \mathbf{r}_t = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^s} q^{s}_{\cdot t} \mathbf{r}^t \tag{1.7}$$

The determinants $|g_{st}|$ and $|G_{st}|$ are denoted by g and G. The volume elements in v- and V-volumes are determined by the expressions

$$d\tau_0 = \sqrt{g}\, dq^1 dq^2 dq^3, \qquad d\tau = \sqrt{G}\, dq^1 dq^2 dq^3$$

and it follows from the law of conservation of mass that

$$\rho_0 \sqrt{g} = \rho \sqrt{G} \tag{1.8}$$

where ρ_0 and ρ are the densities in the initial and final states.

The transpose of a tensor Q is denoted by Q^*, e.g.,

$$(\nabla \mathbf{a})^* = \mathbf{r}^t \mathbf{r}^s \nabla_s a_t \tag{1.9}$$

The differential of a vector $\mathbf{a}$ is expressed in one of its forms as

$$d\mathbf{a} = d\mathbf{r} \cdot \nabla \mathbf{a} = (\nabla \mathbf{a})^* \cdot d\mathbf{r} = d\mathbf{R} \cdot \nabla^\circ \mathbf{a} = (\nabla^\circ \mathbf{a})^* \cdot d\mathbf{R} \tag{1.10}$$

2. Measures of deformation. In the geometry of the v-volume we compute the tensor gradient of the radius vector $\mathbf{R}$ of a point in the V-volume, and also the transpose of this tensor. In accordance with Equations (1.1),(1.5) and (1.9)

$$\nabla \mathbf{R} = \mathbf{r}^s \mathbf{R}_s, \qquad (\nabla \mathbf{R})^* = \mathbf{R}_s \mathbf{r}^s \tag{2.1}$$

From this the two tensor measures of deformation which will be used below are defined

$$G^\times = \nabla \mathbf{R} \cdot (\nabla \mathbf{R})^* = \mathbf{r}^s \mathbf{R}_s \cdot \mathbf{R}_k \mathbf{r}^k = G_{sk} \mathbf{r}^s \mathbf{r}^k \tag{2.2}$$

$$M = (\nabla \mathbf{R})^* \cdot \nabla \mathbf{R} = \mathbf{R}_s \mathbf{r}^s \cdot \mathbf{r}^k \mathbf{R}_k = g^{sk} \mathbf{R}_s \mathbf{R}_k \tag{2.3}$$

The tensor $G^\times$ is defined in the metric of the v-volume, M in the V-volume. The invariants of these tensors are equal, since the eigenvalues of the symmetric matrices of the products AB and BA of matrices A and B are equal. We shall limit ourselves here to the introduction of these two measures of deformation.

We note that the covariant components in the v-metric of the measure of degormation $G^\times$ are equal to the covariant components of the metric tensor G of the V-volume. However, these are different tensors. Thus, the contravariant components of the measure of deformation are computed according to the general rule for transformation from covariant to contravariant components in the v-metric

$$G^{\times st} = g^{sk} g^{tr} G_{kr} \neq G^{st} \tag{2.4}$$

Analogously,

$$M^{st} = g^{st}, \qquad M_{st} = G_{sk} G_{tr} g^{kr} \neq g_{st}$$

Setting $d\mathbf{r} = \mathbf{e} \| d\mathbf{r} \| = \mathbf{e}\, ds$

$$\sqrt{d\mathbf{R} \cdot d\mathbf{R}} = dS = \sqrt{d\mathbf{r} \cdot \nabla \mathbf{R} \cdot (\nabla \mathbf{R})^* \cdot d\mathbf{r}} = ds \sqrt{\mathbf{e} \cdot G^\times \cdot \mathbf{e}}$$

so that the change of a line element of the v-volume is determined by the measure of deformation $G^\times$.

In the v-volume consider the elementary tetrahedron $MA_1A_2A_3$ having edges $MA_s = \mathbf{r}_s dq^s$ extending from the vertex M. The oriented area $A_1A_2A_3$ is given by the vector $\frac{1}{2}\mathbf{n}do$, where $\mathbf{n}$ is the unit vector normal to the area. Then

$$\mathbf{n}do = {}^1/_2 (\mathbf{r}_1 \times \mathbf{r}_2 dq^1 dq^2 + \mathbf{r}_2 \times \mathbf{r}_3 dq^2 dq^3 + \mathbf{r}_3 \times \mathbf{r}_1 dq^3 dq^1) = \\ = \sqrt{g}\,(\mathbf{r}^1 dq^2 \cdot dq^3 + \mathbf{r}^2 dq^3 dq^1 + \mathbf{r}^3 dq^1 dq^2)$$

so that

$$\sqrt{g} dq^1 dq^2 = n_3 do, \quad \sqrt{g} dq^2 dq^3 = n_1 do, \quad \sqrt{g} dq^3 dq^1 = n_2 do, \quad n_s = \mathbf{n} \cdot \mathbf{r}_s$$

In exactly the same way, for the V-volume

$$\mathbf{N}d0 = \sqrt{G}\,(\mathbf{R}^1 dq^2 dq^3 + \mathbf{R}^2 dq^3 dq^1 + \mathbf{R}^3 dq^1 dq^2)$$

$$\sqrt{G} dq^1 dq^2 = N_3^\times d0, \quad \sqrt{G} dq^2 dq^3 = N_1^\times d0, \quad \sqrt{G} dq^3 dq^1 = N_2^\times d0, \quad N_s^\times = \mathbf{N} \cdot \mathbf{R}_s$$

so that

$$\mathbf{N}d0 = \sqrt{G/g}\,\mathbf{R}^s n_s do, \qquad \mathbf{n}do = \sqrt{g/G}\,\mathbf{r}^s N_s^\circ do \tag{2.5}$$

and further

$$\frac{d0}{do} = \left(\frac{G}{g} G^{sk} n_s n_k\right)^{1/2}, \qquad \frac{do}{d0} = \left(\frac{g}{G}\ \mathbf{N} \cdot \boldsymbol{M} \cdot \mathbf{N}\right)^{1/2} \tag{2.6}$$

This last equation determines the geometric significance of the measure of deformation $\boldsymbol{M}$.

The following expression for the unit vector $\mathbf{N}$ normal to the surface S will be applied many times below

$$\mathbf{N}\sqrt{G^{sk} n_s n_k} = \mathbf{R}^s n_s = \mathbf{R}^s \mathbf{r}_s \cdot n = \nabla^\circ \mathbf{r} \cdot \mathbf{n} \tag{2.7}$$

where $\nabla^\circ \mathbf{r}$ is the gradient of the radius vector of the v-volume computed in the metric of the V-volume; this is the inverse of the tensor $\nabla\mathbf{R}$

$$\nabla^\circ \mathbf{r} \cdot \nabla \mathbf{R} = \mathbf{R}^s \mathbf{r}_s \cdot \mathbf{r}^k \mathbf{R}_k = \mathbf{R}^s \mathbf{R}_s = \boldsymbol{G}, \quad \nabla^\circ \mathbf{r} = (\nabla \mathbf{R})^{-1} \tag{2.8}$$

With the aid of the tensors $\nabla^\circ \mathbf{r}$ and $(\nabla^\circ \mathbf{r})^*$, the measures of deformation

$$\mathbf{g}^\times = \nabla^\circ \mathbf{r} \cdot (\nabla^\circ \mathbf{r})^* = g_{sk} \mathbf{R}^s \mathbf{R}^k = \mathbf{M}^{-1}, \qquad \mathbf{m} = (\nabla^\circ \mathbf{r})^* \cdot \nabla^\circ \mathbf{r} = G^{sk} \mathbf{r}_s \mathbf{r}_k = (\mathbf{G}^\times)^{-1}$$

are defined as in (2.2) and (2.3).

It is obvious from these relations that the eigenvalues of the tensors $\boldsymbol{G}^\times$ and $\boldsymbol{M}$, which are equal to each other, $G_s^\times = M_s$ are equal to the reciprocals of the eigenvalues $g_s^\times = m_s$ of the tensors $\boldsymbol{g}$ and $\boldsymbol{m}$

$$G_s^\times = M_s = (g_s^\times)^{-1} = m_s^{-1}$$

From this fact the equations which relate the principal invariants of these tensors follow

$$I_1(\boldsymbol{G}^\times) = \frac{1}{I_3(\boldsymbol{g}^\times)} I_2(\boldsymbol{g}^\times), \quad I_2(\boldsymbol{G}^\times) = \frac{1}{I_3(\boldsymbol{g}^\times)} I_1(\boldsymbol{g}^\times), \quad I_3(\boldsymbol{G}^\times) = \frac{1}{I_3(\boldsymbol{g}^\times)} \tag{2.9}$$

But, according to (2.2) and (2.8)

$$I_1(\boldsymbol{G}^\times) = G_{sk} \mathbf{r}^s \cdot \mathbf{r}^k = g^{sk} G_{sk}, \qquad I_1(\boldsymbol{g}^\times) = g_{sk} \mathbf{R}^s \cdot \mathbf{R}^k = g_{sk} G^{sk}$$

and, defining $I_3(\boldsymbol{G}^\times)$ as the square of the volume ratio $d\tau/d\tau_0$, we have

$$I_1(\boldsymbol{G}^\times) = g^{sk} G_{sk}, \qquad I_2(\boldsymbol{G}^\times) = \frac{G}{g} g_{sk} G^{sk}, \qquad I_3(\boldsymbol{G}^\times) = \frac{G}{g} \tag{2.10}$$

3. The stress tensor. This symmetric tensor $\boldsymbol{T} = \boldsymbol{T}^*$ of second order is usually defined in the V-volume by its contravariant or mixed components

$$\boldsymbol{T} = \tau^{\circ st}\mathbf{R}_s\mathbf{R}_t = \tau_t^{\circ s}\mathbf{R}_s\mathbf{R}^t \tag{3.1}$$

Its product with the oriented area vector NdO in the V-volume determines the force acting on this area $\mathbf{t}_N dO$

$$\mathbf{t}_N = \mathbf{N}\cdot\boldsymbol{T} = \boldsymbol{T}\cdot\mathbf{N} = \tau^{\circ st}N_s\mathbf{R}_t = \tau_t^{\circ s}N_s\mathbf{R}^t \tag{3.2}$$

The equation of equilibrium of a volume of the continuous medium is written in the form

$$\nabla^\circ\cdot\boldsymbol{T} + \rho\mathbf{K} = 0 \tag{3.3}$$

where $\mathbf{K}$ is the body force vector (per unit mass). From (1.7) and (1.8) the last equation may be represented in the forms

$$\frac{\partial}{\partial q^s}\sqrt{G}\tau^{\circ st}\mathbf{R}_t + \rho_0\sqrt{g}\mathbf{K} = 0, \qquad \frac{\partial}{\partial q^s}\sqrt{G}\tau_t^{\circ s}\mathbf{R}^t + \rho_0\sqrt{g}\mathbf{K} = 0 \tag{3.4}$$

The equation of equilibrium on the surface S is adjoined to the above which, from Equations (2.7) and (3.2), is written in the form

$$\mathbf{F}\sqrt{G^{sk}n_sn_k} = \tau^{\circ st}n_s\mathbf{R}_t = \tau_t^{0s}n_s\mathbf{R}^t \tag{3.5}$$

where $\mathbf{F}$ is the external surface force per unit area of S .

The virtual work per unit v-volume of the external forces $\mathbf{K}$ and $\mathbf{F}$ can be represented as

$$\delta' A_{(e)} = {}^1/_2\sqrt{G/g}\,\tau^{\circ st}\delta G_{st} \tag{3.6}$$

In a perfectly elastic body this quantity is equated to the variation of the strain energy dentity A , which is equal to the variation of the internal energy density of the body for an adiabatic process of deformation and to the free energy in an isothermal process. The specification of this quantity in terms of the components of the measure of deformation $G^\times$ and the temperature (in an adiabatic process) or the entropy (in an isothermal one) determines the equation of state of a perfectly elastic body

$$\frac{1}{2}\sqrt{\frac{G}{g}}\,\tau^{\circ st} = \frac{\partial A}{\partial G_{st}} \tag{3.7}$$

4. The equation of state of a perfectly elastic isotropic body. In an isotropic body the strain energy density A depends on the invariants of one or the other measure of deformation. These invariants are expressed in terms of each other in (2.9). Let us assume that A is given in terms of the invariants $I_k(G^\times) = I_k$.

Then, according to Equation (3.7),

$$\frac{1}{2}\sqrt{\frac{G}{g}}\,\tau^{0st} = \sum_{k=1}^{3}\frac{\partial A}{\partial I_k}\frac{\partial I_k}{\partial G_{st}}$$

From (2.10) we have

$$\frac{\partial I_1}{\partial G_{st}} = g^{st}, \qquad \frac{\partial I_3}{\partial G_{st}} = \frac{1}{g}\frac{\partial G}{\partial G_{st}} = I_3G^{st}$$

$$\frac{\partial I_2}{\partial G_{st}} = I_1g^{st} - (G^\times)^{st} = I_1g^{st} - g^{sq}g^{tr}G_{qr}$$

(without going into the details of the derivation of these from formulas which have been given). The equation of state of a perfectly elastic isotropic body can then be written in the form

$$^1/_2 \sqrt{G/g}\, \tau^{\circ st} = c^{(0)} g^{st} - c^{(1)} g^{sq} g^{tr} G_{qr} + c^{(-1)} G^{st} \tag{4.1}$$

where the "generalized moduli of elasticity" are denoted by $c^{(s)}$

$$c^{(0)} = \frac{\partial A}{\partial I_1} + I_1 \frac{\partial A}{\partial I_2}, \qquad c^{(1)} = \frac{\partial A}{\partial I_2}, \qquad c^{(-1)} = I_3 \frac{\partial A}{\partial I_3} \tag{4.2}$$

Noting that from (2.3)

$$M^2 = g^{sq}\mathbf{R}_s\mathbf{R}_q \cdot g^{rt}\mathbf{R}_r\mathbf{R}_t = g^{sq}g^{tr}G_{qr}\mathbf{R}_s\mathbf{R}_t$$

and recalling (3.1) and (1.4), we arrive at the equation of state in Finger's form.

$$^1/_2 \sqrt{G/g}\, \boldsymbol{T} = c^{(0)} \boldsymbol{M} - c^{(1)} \boldsymbol{M}^2 + c^{(-1)} \boldsymbol{G} \tag{4.3}$$

It is natural to express the tensor $\boldsymbol{T}$ in terms of tensors defined in the V-volume. It can also be related to the Almansi measure of deformation $\boldsymbol{g}^\times$; for from (2.8)

$$^1/_2 \sqrt{G/g}\, \boldsymbol{T} = c^{(0)} (\boldsymbol{g}^\times)^{-1} - c^{(1)} (\boldsymbol{g}^\times)^{-2} + c^{-1}\boldsymbol{G} = e^{(0)}\boldsymbol{G} - e^{(1)}\boldsymbol{g}^\times + e^{(2)}\boldsymbol{g}^{\times 2} \tag{4.4}$$

The negative powers of the tensor $\boldsymbol{g}^\times$ h re are expressed in terms of positive powers (e.g. with the aid of the Cayley-Hamilton theorem). The generalized moduli $e^{(s)}$ are determined by Equations (4.5)

$$e^{(0)} = -I_3' \frac{\partial A}{\partial I_3'}, \qquad e^{(1)} = \frac{\partial A}{\partial I_1'} + I_1' \frac{\partial A}{\partial I_2'}, \qquad e^{(2)} = \frac{\partial A}{\partial I_2'}, \qquad (I_k' = I_k(\boldsymbol{g}^\times))$$

which are obtained by replacing the variables I_k by the I_k' with the aid of (2.9).

E x a m p l e . For a bar loaded by forces which are uniformly distributed on its ends and are directed along the axis of the bar Oa_3, the point transformation from the initial state to the final one (from the v-volume to the V-volume) is given by Formulas

$$x_1 = \alpha_1 a_1, \quad x_2 = \alpha_2 a_2, \quad x_3 = \alpha_3 a_3 \quad (\alpha_s = \text{const})$$

where the a_s are the Cartesian coordinates in the v-volume, which play the role of the material coordinates. Here $\boldsymbol{g} = \boldsymbol{E}$, the unit tensor

$$G_{ss} = \alpha_s^2, \qquad G^{ss} = \alpha_s^{-2}, \qquad G^{sk} = G_{sk} = 0 \quad \text{for } s \neq k$$

so that

$$g = 1, \quad G = I_3 = \alpha_1^2\alpha_2^2\alpha_3^2, \quad I_1 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2, \quad I_2 = \alpha_1^2\alpha_2^2 + \alpha_2^2\alpha_3^2 + \alpha_3^2\alpha_1^2$$

and, from (4.1) and (4.2) the nonzero contravariant components of the tensor are

$$^1/_2 \alpha_1\alpha_2\alpha_3 \tau^{ss} = c^0 - c^{(1)}\alpha_s^2 + c^{(-1)}\alpha_s^{-2} \qquad (s = 1, 2, 3)$$

The equation of equilibrium (3.3) is satisfied, since $\mathbf{T}$ is a constant tensor. From the condition that surface tractions are absent on the lateral surface, it follows that $\tau^{011} = \tau^{022} = 0$ throughout the volume, so that $\alpha_1 = \alpha_2$ and

$$c^{(0)} - c^{(1)}\alpha_1^2 + c^{(-1)}\alpha_1^{-2} = 0, \qquad ^1/_2 \alpha_1^2\alpha_3 \tau^{033} = c^{(0)} - c^{(1)}\alpha_3^2 + c^{(-1)}\alpha_3^{-2} \tag{4.6}$$

The physical component t^{33} of the tensor is equal in this case to $\alpha_3^2\tau^{033}$; therefore, the axial force $Q = St^{33} = S_0\alpha_1^2 t^{33}$, where S_0 is the area of the original cross section of the bar, is expressed in the form

$$Q = 2\left(c^{(0)}\alpha_3 - c^{(1)}\alpha_3^3 + \frac{c^{(-1)}}{\alpha_3} \right) S_0 \tag{4.7}$$

We shall specify the expression for the strain energy density in a form which is equivalent (*) to the expression in the linear theory of elasticity

$$A = {}^1/_8 (\lambda + 2\mu) I_1^2 - {}^1/_4 (3\lambda + 2\mu) I_1 - {}^1/_2 \mu I_2 \qquad (4.8)$$

where λ and μ are constants. Then

$$c^{(0)} = {}^1/_4 (\lambda I_1 - 3\lambda - 2\mu), \qquad c^{(1)} = -{}^1/_2\mu, \qquad c^{(-1)} = 0 \qquad (I_1 = 2\alpha_1^2 + \alpha_3^2)$$

so that from (4.6)

$$c^{(0)} - c^{(1)}\alpha_1^2 = \frac{1}{4}(\lambda I_1 - 3\lambda - 2\mu) + \frac{\mu}{2}\alpha_1^2 = 0, \quad t^{33} = \mu \frac{\alpha_3}{\alpha_1^2}(\alpha_3^2 - \alpha_1^2) \qquad (4.9)$$

From the first equation we find the relation connecting α_1^2 and α_3^2

$$\alpha_1^2 + \nu\alpha_3^2 = \nu + 1 \qquad \left(\nu = \frac{\lambda}{2(\lambda + \mu)}\right) \qquad (4.10)$$

and from (4.7) and (4.9) we find

$$Q = {}^1/_2 ES_0\alpha_3(\alpha_3^2 - 1), \qquad E = 2\mu(1 + \nu) \qquad (4.11)$$

or, noting that $\alpha_3 = 1 + \delta_3$, where δ_3 is the unit axial extension,

$$Q = ES_0\delta_3(1 + \delta_3)(1 + {}^1/_2\delta_3)$$

This is the nonlinear law of behavior of a stretched specimen under the assumption that the strain energy density is defined according to (4.8). It reduces to the linear Hooke's law for $\delta_3 \ll 1$. The notation λ, μ, ν, and E was, of course, chosen by analogy with the linear theory. Further on, in Sections 8 and 9, Equations (4.9) to (4.11) will be applied in the case of compression; then $\alpha_3 < 1$ and $\alpha_1 > 1$.

5. Superposition of deformations. A displacement defined by the vector $\mu\mathbf{w}$ is given to the points of the V-volume, where μ is a small parameter. In the calculations given below, only terms of first degree in this parameter are taken into account. The radius vector of a point in the resulting V'-volume which is bounded by the surface S' is equal to

$$\mathbf{R}' = \mathbf{R} + \mu\mathbf{w} \qquad (5.1)$$

and the vector basis is given by the triad of vectors (**)

$$\mathbf{R}_s' = \mathbf{R}_s + \mu\frac{\partial\mathbf{w}}{\partial q^s} = \mathbf{R}_s + \mu\mathbf{R}^t\nabla_s w_t \qquad (5.2)$$

The covariant components of the metric tensor G' in the V'-volume are, therefore, equal to

$$G_{sk}' = R_s' \cdot R_k' = G_{sk} + \mu(\nabla_s w_k + \nabla_k w_s) = G_{sk} + 2\mu\varepsilon_{sk} \qquad (5.3)$$

where ε_{sk} are the covariant components of a linear strain tensor

$$\varepsilon = \operatorname{def}\mathbf{w} = {}^1/_2[\nabla\mathbf{w} + (\nabla\mathbf{w})^*] = {}^1/_2\mathbf{R}^s\mathbf{R}^k(\nabla_s w_k + \nabla_k w_s)$$

The covariant components of the tensor $\mathbf{G}'$ are determined from the relation

$$G_k'^s = G'^{sr}G_{rk}' = (G^{sr} + \mu q^{sr})(G_{rk} + 2\mu\varepsilon_{rk}) = G_k^s + \mu q^{sr}G_{rk} + 2\mu G^{sr}\varepsilon_{rk}$$

so that, inasmuch as $G_k'^s = G_k^s$

$$q^{sr}G_{rk} + 2G^{sr}\varepsilon_{rk} = 0, \qquad G'^{tr} = G^{tr} - 2\mu G^{sr}G^{kt}\varepsilon_{sk} \qquad (5.4)$$

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*) In the linear theory the energy is given in terms of the invariants of the strain tensor and not those of the measure of deformation.

**) In all that follows, the calculations are carried out in the metric of the V-volume; the index ° is, therefore, omitted.

The vectors of the associated basis are now constructed in accordance with (1.2)

$$\mathbf{R}'^t = G'^{tr}\mathbf{R}_r' = \mathbf{R}^t - 2\mu\mathbf{R}^q G^{kt}\varepsilon_{qk} + G^{kt}\mathbf{R}^q\nabla_k w_q = \mathbf{R}^t - G^{\kappa t}\mathbf{R}^q\nabla_q w_k$$

To the same order of accuracy, the metric tensor of the V'-volume turns out to be

$$G' = \mathbf{R}_t'\mathbf{R}'^t = \mathbf{R}_t\mathbf{R}^t + \mu(\mathbf{R}^m\mathbf{R}^t\nabla_t w_m - \mathbf{R}^t\mathbf{R}^m\nabla_m w_t) = \mathbf{R}_t\mathbf{R}^t = G \quad (5.5)$$

The determinant G' is computed most simply from the relation

$$G' = G + \mu\left(\frac{\partial G}{\partial\mu}\right)_{\mu=0} = G + \mu\frac{\partial G}{\partial G_{st}}\frac{\partial G_{st}'}{\partial\mu} = G + 2\mu G G^{st}\varepsilon_{st}$$

so that

$$G' = G(1 + 2\mu\vartheta), \qquad \sqrt{G'} = \sqrt{G}(1 + \mu\vartheta), \qquad \vartheta = G^{st}\varepsilon_{st} \quad (5.6)$$

where ϑ is the first invariant of ε .

The expressions for the principal invariants of the measure of deformation $G^{\times\prime}$ are computed from (2.10),(5.3),(5.4) and (5.6)

$$I_1(G^{\times\prime}) = I_1 + 2\mu g^{sk}\varepsilon_{sk}, \qquad I_3(G^{\times\prime}) = I_3(1 + 2\mu\nu)$$

$$I_2(G^{\times\prime}) = I_2 + 2\mu(\vartheta I_2 - I_3 g_{sk}G^{st}G^{kq}\varepsilon_{tq}) \quad (5.7)$$

so that the volume element on the V'-volume is

$$d\tau' = (1 + \mu\vartheta)\,d\tau, \qquad d\tau = \sqrt{G}dq^1dq^2dq^3$$

The vector of the oriented area can be written in the following form, in accordance with (2.5),(5.5) and (5.6):

$$\mathbf{N}'dO' = \sqrt{\frac{G'}{G}}\,\mathbf{R}'^s N_s dO = \{\mathbf{N} + \mu[\vartheta\mathbf{N} - N^t\mathbf{R}^q\nabla_q w_t]\}\,dO$$

From this we find that

$$\frac{dO'}{dO} = 1 + \mu(\vartheta - N^r N^q\nabla_r w_q), \qquad \mathbf{N}' = \mathbf{N} + \mu(\mathbf{N}N^r N^t - N^t\mathbf{R}^r)\nabla_r w_t \quad (5.8)$$

Recalling the definition of the tensor $\nabla\mathbf{w}$

$$\nabla\mathbf{w} = \mathbf{R}^r\mathbf{R}^t\nabla_r w_t \quad (5.9)$$

we may also write (5.8) in the form

$$\mathbf{N}' = \mathbf{N} + \mu(\mathbf{N}\mathbf{N}\cdot\nabla\mathbf{w}\cdot\mathbf{N} - \nabla\mathbf{w}\cdot\mathbf{N}) = \mathbf{N} + \mu\mathbf{N}\times[\mathbf{N}\times(\nabla\mathbf{w}\cdot\mathbf{N})] \quad (5.10)$$

As was to be expected, the vector $\mathbf{N}'$ differes from $\mathbf{N}$ by a vector which lies in the plane of the area. Let us now also develop an expression for the measure of deformation M' which will be required below. From (5.5), (2.3) and (5.8) we have

$$M' = g^{sk}\mathbf{R}_s'\mathbf{R}_k' = M + \mu((\nabla\mathbf{w})^*\cdot M + M\cdot\dot{\nabla}\mathbf{w}) \quad (5.11)$$

whence we also have

$$M'^2 = M^2 + \mu(M^2\cdot\nabla\mathbf{w} + 2M\cdot\varepsilon\cdot M + (\nabla\mathbf{w})^*\cdot M^2) \quad (5.12)$$

6. The stress tensor and the equilibrium equations. The difference of the contravariant components of the stress tensors in the V'- and V-volumes

$$\tau'^{st} - \tau^{st} = \mu p^{st} \quad (6.1)$$

will be considered. The stress tensor $\boldsymbol{T}'$ in the V'-volume and in the metric of that volume is then determined from (5.2) by Equation

$$\boldsymbol{T}' = \tau'^{st}\mathbf{R}_s'\mathbf{R}_t' = \boldsymbol{T} + \mu p^{st}\mathbf{R}_s\mathbf{R}_t + \mu\tau^{st}(\mathbf{R}^q\mathbf{R}_t\nabla_s w_q + \mathbf{R}_s\mathbf{R}^q\nabla_t w_q) = \\ = \boldsymbol{T} + \mu(p^{st}\mathbf{R}_s\mathbf{R}_t + \boldsymbol{T}\cdot\nabla\mathbf{w} + (\nabla\mathbf{w})^*\cdot\boldsymbol{T}) = \boldsymbol{T} + \mu\boldsymbol{S} \tag{6.2}$$

where S is the symmetric tensor of the additional stresses

$$S = p^{st}\mathbf{R}_s\mathbf{R}_t + \boldsymbol{T}\cdot\nabla\mathbf{w} + (\nabla\mathbf{w})^*\cdot\boldsymbol{T} \tag{6.3}$$

The equation of equilibrium in the V'-volume is written in the form

$$\operatorname{div}'\mathbf{T}' + \rho'\mathbf{K}' = 0$$

Here $\rho'\mathbf{K}'$ is the body force per unit volume and $\mathbf{K}'$ is the body force per unit mass

$$\mathbf{K}' = \mathbf{K} + \mu\mathbf{k} \tag{6.4}$$

and the density ρ' in the V'-volume is determined, according to Equations (1.8) and (5.6), by Equation

$$\rho'\sqrt{G'} = \rho\sqrt{G} = \rho_0\sqrt{g} \tag{6.5}$$

Referring to (3.4) and (6.1), we have

$$\frac{1}{\sqrt{G'}}\frac{\partial}{\partial q^s}\left(\sqrt{G'}\tau'^{st}\mathbf{R}_t'\right) + \rho'(\mathbf{K} + \mu\mathbf{k}) = 0$$

Substituting the values of G', τ'^{st}, $\boldsymbol{R}_t'$ from (6.1),(5.6) and (5.2), we find

$$\frac{\partial}{\partial q^s}\left[\sqrt{G}(1 + \mu\vartheta)(\tau^{st} + \mu p^{st})(\mathbf{R}_t + \mu\mathbf{R}^q\nabla_t w_q)\right] + \rho_0\sqrt{g}(\mathbf{K} + \mu\mathbf{k}) = 0$$

or, from (3.4)

$$\frac{\partial}{\partial q^s}\sqrt{G}(\vartheta\tau^{st}\mathbf{R}_t + \tau^{st}\nabla_t w_q\mathbf{R}^q + p^{st}\mathbf{R}_t) + \rho_0\sqrt{g}\mathbf{k} = 0$$

But from (5.9) and (1.7)

$$\boldsymbol{T}\cdot\nabla\mathbf{w} = \mathbf{R}_s\mathbf{R}^q\tau^{st}\nabla_t w_q, \qquad \operatorname{div}\boldsymbol{T}\cdot\nabla\mathbf{w} = \frac{1}{\sqrt{G}}\frac{\partial}{\partial q^s}\sqrt{G}\tau^{st}\nabla_t w_q\mathbf{R}^q$$

and the preceding equation may be written in the form

$$\operatorname{div}(p^{st}\mathbf{R}_s\mathbf{R}_t + \vartheta\boldsymbol{T} + \boldsymbol{T}\cdot\nabla\mathbf{w}) + \rho\mathbf{k} = 0$$

Therefore, by introducing the nonsymmetric second-order tensor

$$\Sigma = \boldsymbol{S} + \vartheta\boldsymbol{T} - (\nabla\mathbf{w})^*\cdot\boldsymbol{T} \tag{6.6}$$

the equation of equilibrium may be put into the form

$$\operatorname{div}\Sigma + \rho\mathbf{k} = 0 \tag{6.7}$$

We now turn to the equation of equilibrium on the surface S'.

The surface force per unit area is determined by Equation

$$\mathbf{F}' = \mathbf{F}\frac{d0}{d0'} + \mu\mathbf{f}$$

where $\mathbf{F}$ is the surface traction on S. Then from (3.2)

$$\mathbf{F}\frac{d0}{d0'} + \mu\mathbf{f} = \mathbf{N}'\cdot\boldsymbol{T}' = \mathbf{N}\cdot\boldsymbol{T} + (\mathbf{N}' - \mathbf{N})\cdot\boldsymbol{T} + \mathbf{N}\cdot(\boldsymbol{T}' - \boldsymbol{T})$$

so that, referring to Equations (5.8),(5.10), (6.2) and (3.2), we have

$$\mathbf{F}\,[1-\mu(\vartheta+\mathbf{N}\cdot\nabla\mathbf{w}\cdot\mathbf{N})]+\mu\mathbf{f}=\mathbf{F}+\mu\,[\mathbf{F}\mathbf{N}\cdot\nabla w\cdot\mathbf{N}-(\nabla\mathbf{w}\cdot\mathbf{N})\cdot\boldsymbol{T}+\mathbf{N}\cdot\boldsymbol{S}]$$

Remembering now that $\nabla\mathbf{w}\cdot\mathbf{N}=\mathbf{N}\cdot(\nabla\mathbf{w})^*$ from (6.6), we arrive at Equation

$$\mathbf{N}\cdot[\boldsymbol{S}+\vartheta\boldsymbol{T}-(\nabla\mathbf{w})^*\cdot\boldsymbol{T}]=\mathbf{N}\cdot\Sigma=\mathbf{f} \tag{6.8}$$

Thus, the equation of equilibrium and the boundary condition are expressed in terms of the same nonsymmetric second-order tensor $\boldsymbol{\Sigma}$.

7. The equation of state. The equation of state for the tensor $\boldsymbol{T}'$ is written in the form (4.3)

$$\tfrac{1}{2}\sqrt{G'/g}\,\boldsymbol{T}'=c^{(0)\prime}\boldsymbol{M}'-c^{(1)\prime}\boldsymbol{M}'^2+c^{(-1)\prime}\boldsymbol{G}'$$

Referring to (4.3),(5.5) and (5.6), we can write this equation in the form

$$\tfrac{1}{2}\sqrt{G/g}\,[(\boldsymbol{T}'-\boldsymbol{T})+\mu\vartheta\boldsymbol{T}]=$$
$$=\mu(b^{(0)}\boldsymbol{M}-b^{(1)}\boldsymbol{M}+b^{(-1)}\boldsymbol{G})+c^{(0)}(\boldsymbol{M}'-\boldsymbol{M})-c^{(1)}[\boldsymbol{M}'^2-\boldsymbol{M}^2)$$
$$\mu b^{(0)}=c^{(0)\prime}-c^{(0)},\qquad \mu b^{(1)}=c^{(1)\prime}-c^{(1)},\qquad \mu b^{(-1)}=c^{(-1)\prime}-c^{(-1)}$$

From Equations (6.2),(6.6) and (5.12), we now obtain

$$\tfrac{1}{2}\sqrt{G/g}\,\Sigma=b^{(0)}\boldsymbol{M}-b^{(1)}\boldsymbol{M}^2+b^{(-1)}\boldsymbol{G}+c^{(0)}\boldsymbol{M}\cdot\nabla\mathbf{w}-$$
$$-c^{(1)}(\boldsymbol{M}^2\cdot\nabla w+2\boldsymbol{M}\cdot\varepsilon\cdot\boldsymbol{M})-c^{(-1)}(\nabla\mathbf{w})^* \tag{7.1}$$

The expressions for the "moduli" $b^{(s)}$ remain to be formulated

$$b^{(s)}=\frac{1}{\mu}\frac{\partial c^{(s)}}{\partial I_k}(I_k'-I_k)=\frac{\partial c^{(s)}}{\partial I_k}i_{(k)}$$

Here, according to (5.7)

$$i_{(1)}=2g^{rt}\varepsilon_{rt},\qquad i_{(2)}=2(\vartheta I_2-I_3g_{sk}G^{st}G^{kq}\varepsilon_{tq}),\qquad i_{(3)}=2I_3\vartheta \tag{7.2}$$

and with the notation $b^{(rs)}=\dfrac{\partial^2 A}{\partial I_r\partial I_s}$ we have

$$b^{(0)}=i_k(b^{(1k)}+I_1b^{(2k)})+i_{(1)}c^{(1)},b^{(1)}=i_{(k)}b^{(2k)},\ b^{(-1)}=i_{(k)}I_3b^{(3k)}+i^{(3)}\frac{c^{(-1)}}{I_3} \tag{7.3}$$

By means of Equation (7.1) the tensor $\boldsymbol{\Sigma}$ is represented in the form of a linear differential operator of first order on the vector $\mathbf{w}$. Substitution of this operator into the equation of equilibrium (6.7) and the boundary condition (6.8) leads to a linear system of three second-order differential equations in the V-volume with linear boundary conditions on the surface S in the absence of additional body forces and surface tractions ($\mathbf{k}=0$, $\mathbf{f}=0$). This linear boundary value problem is homogeneous and the question is to determine the bifurcation values of the parameters of the initial state, e.g. the load parameters $\mathbf{K}$ or $\mathbf{F}$, for which nontrivial solutions ($\mathbf{w}\neq 0$) of the problem exist.

8. The compressed bar. If the strain energy density is given in the form (4.8), only the modulus $b^{(0)}=\tfrac{1}{2}\lambda g^{st}\varepsilon_{st}$, is nonzero. Taking into account that (4.9) is satisfied in the initial state, we can write the expression for the tensor $\boldsymbol{\Sigma}$ in the form

$$\alpha_1^2\alpha_3\,\Sigma=\lambda g^{st}\varepsilon_{st}\boldsymbol{M}+\mu\{-\alpha_1^2\boldsymbol{M}\cdot\nabla\mathbf{w}+\boldsymbol{M}^2\cdot\nabla\mathbf{w}+\boldsymbol{M}\cdot[\nabla\mathbf{w}+(\nabla\mathbf{w})^*]\cdot\boldsymbol{M}\}$$

where

$$M = \mathbf{R}_s\mathbf{R}_s, \qquad \nabla\mathbf{w} = \mathbf{R}^m\mathbf{R}^n\partial_m w_n, \qquad \partial_m = \frac{\partial}{\partial a_m}, \qquad w_n = \mathbf{R}_n\cdot\mathbf{w}$$

We then arrive at the following expressions for contravariant components of Σ :

$$\alpha_1^2\alpha_3\sigma^{11} = \lambda\vartheta + 2\mu\partial_1 w_1, \qquad \alpha_1^2\alpha_3\sigma^{22} = \lambda\vartheta + 2\mu\partial_2 w_2$$
$$\alpha_1^2\alpha_3\sigma^{12} = \alpha_1^2\alpha_3\sigma^{21} = \mu(\partial_1 w_2 + \partial_2 w_1) \quad (\vartheta = \partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3) \tag{8.1}$$
$$\alpha_1^2\alpha_3\sigma^{13} = \mu(\partial_1 w_3 + \partial_3 w_1), \qquad \alpha_1^2\alpha_2\sigma^{23} = \mu(\partial_2 w_3 + \partial_3 w_2)$$

These are formulas which are analogous to Hook's law for the linear theory of elasticity; in contrast to this

$$\alpha_1^2\alpha_3\sigma^{31} = \mu\left(\partial_1 w_3 + \frac{\alpha_3^2}{\alpha_1^2}\partial_3 w_1\right), \qquad \alpha_1^2\alpha_3\sigma^{32} = \mu\left(\partial_2 w_3 + \frac{\alpha_3^2}{\alpha_1^2}\partial_3 w_2\right)$$
$$\alpha_1^2\alpha_3\sigma^{33} = \lambda\vartheta + 2\mu\partial_3 w_3 + \mu\left(1 - \frac{\alpha_1^2}{\alpha_3^2}\right)\partial_3 w_3 \tag{8.2}$$

The equation of equilibrium (6.7) are written in the form (8.3)

$$\partial_1\sigma^{11} + \partial_2\sigma^{21} + \partial_3\sigma^{31} = 0, \quad \partial_1\sigma^{12} + \partial_2\sigma^{22} + \partial_3\sigma^{32} = 0, \quad \partial_1\sigma^{13} + \partial_2\sigma^{23} + \partial_3\sigma^{33} = 0$$

The boundary conditions (6.8) on the lateral surface of the bar are

$$N_1\sigma^{11} + N_2\sigma^{21} = 0, \qquad N_1\sigma^{12} + N_2\sigma^{22} = 0, \qquad N_1\sigma^{13} + N_2\sigma^{23} = 0 \tag{8.4}$$

Here $N_s = \mathbf{N}\cdot\mathbf{R}_s$ are the covariant components of the unit normal vector $\mathbf{N}$, with $N_3 = 0$; on the ends $N_3 \neq 0$, and the boundary conditions on the upper free end $a_3 = l$ are written in the form (for a cantiliver beam)

$$a_3 = l; \quad \sigma^{31} = 0, \quad \sigma^{32} = 0, \quad \sigma^{33} = 0 \tag{8.5}$$

Substituting (8.1) and (8.2) into the equation of equilibrium (8.3), we arrive at a system of linear differential equation for the components of the vector $\mathbf{w}$

$$(\lambda + \mu)\partial_1\vartheta + \mu\left(D^2 w_1 + \frac{\alpha_3^2}{\alpha_1^2}\partial_3^2 w_1\right) = 0$$
$$(\lambda + \mu)\partial_2\vartheta + \mu\left(D^2 w_2 + \frac{\alpha_3^2}{\alpha_1^2}\partial_3^2 w_2\right) = 0 \qquad (D^2 = \partial_1^2 + \partial_2^2) \tag{8.6}$$
$$(\lambda + \mu)\partial_3\vartheta + \mu\left[D^2 w_3 + \left(2 - \frac{\alpha_1^2}{\alpha_3^2}\right)\partial_3^2 w_3\right] = 0$$

which differ from the equations in terms of displacements in the linear theory of elasticity and reduce to them for $\alpha_3 = \alpha_1$.

We shall seek the first group of special solutions in the form

$$w_1 = \partial_1\Phi + w_1^\circ, \quad w_2 = \partial_2\Phi + w_2^\circ, \quad w_3 = k\partial_3\Phi \qquad (k, w_1^\circ, w_2^\circ = \text{const}) \tag{8.7}$$

Substitution into Equations (8.6) leads to Equations

$$\frac{1}{q}D^2\Phi + \left(\frac{1-q}{q}k + \frac{\alpha_3^2}{\alpha_1^2}\right)\partial_3^2\Phi = 0, \quad \left(\frac{1-q}{q} + k\right)D^2\Phi + k\left(\frac{1+q}{q} - \frac{\alpha_1^2}{\alpha_3^2}\right)\partial_3^2\Phi = 0$$
$$q = \frac{1-2\nu}{2(1-\nu)} = \frac{\mu}{\lambda + 2\mu} \tag{8.8}$$

This last quantity decreases monotonously from $\frac{1}{2}$ to 0 as ν increases from 0 to $\frac{1}{2}$. If we now restrict the choice of the constant k by the conditions

$$(1-q)k + q\frac{\alpha_3^2}{\alpha_1^2} = \frac{k}{1-q+kq}\left[1 + q\left(1 - \frac{\alpha_1^2}{\alpha_3^2}\right)\right] = \zeta^2 \tag{8.9}$$

we arrive at the differential equations

$$D^2\Phi_s + \zeta_s^2\partial_3^2\Phi_s = 0 \qquad (s = 1, 2) \tag{8.10}$$

where ζ_s^2 are the roots of Equation

$$\zeta^4 - \zeta^2\left[3 - \frac{\alpha_1^2}{\alpha_3^2} + q\left(\frac{\alpha_3^2}{\alpha_1^2} - 1\right)\right] + \frac{\alpha_3^2}{\alpha_1^2} + \left(\frac{\alpha_3^2}{\alpha_1^2} - 1\right)q = 0 \tag{8.11}$$

The discriminant of this equation may be represented in the form

$$\triangle = \left(1 - \frac{\alpha_3^2}{\alpha_1^2}\right)^2 \left[\frac{\alpha_1^4}{\alpha_3^4} - 2(2-q)\frac{\alpha_1^2}{\alpha_3^2} + q^2\right] \tag{8.12}$$

and the roots of the quadratic in brackets are

$$\left(\frac{\alpha_1^2}{\alpha_3^2}\right)_1 = (\sqrt{1-q} - 1)^2, \qquad \frac{\alpha_1^2}{\alpha_3^2} = (\sqrt{1-q} + 1)^2$$

The quadratic is negative between these roots. For a compressed bar $\alpha_1^2/\alpha_3^2 > 1$, so that for all values of q in the interval $(0, \frac{1}{2})$ the discriminant (8.12) is negative for

$$1 < \alpha_1^2/\alpha_3^2 < (\sqrt{1/_2} + 1)^2 = 2.914$$

and for these values of this ratio, the roots ζ_1^2 and ζ_2^2 are complex conjugates. We seek a second group of particular solutions of the system (8.6) in the form

$$w_1 = \partial_2\Psi, \quad w_2 = -\partial_1\Psi, \quad w_3 = 0 \qquad (\vartheta = 0) \tag{8.13}$$

where Ψ is determined by the differential equation

$$D^2\Psi + \frac{\alpha_3^2}{\alpha_1^2}\Psi = 0 \tag{8.14}$$

The expressions for the components of the tensor $\boldsymbol{E}$ corresponding to the system of particular solutions (8.7) reduce to the following form, taking account of (8.10):

$$\frac{1}{\mu}\alpha_1^2\alpha_3\sigma_s^{mm} = \frac{2q-1}{q-1}\left(\frac{1}{\zeta_s^2}\frac{\alpha_3^2}{\alpha_1^2} - 1\right)D^2\Phi_s + 2\partial_m^2\Phi_s \tag{8.15}$$

$$\frac{1}{\mu}\alpha_1^2\alpha_3\,\sigma_s^{12} = \frac{1}{\mu}\alpha_1^2\alpha_3\sigma_s^{21} = 2\partial_1\partial_2\Phi_s$$

$$\frac{1}{\mu}\alpha_1^2\alpha_3\sigma_s^{m3} = (k_s + 1)\,\partial_3\partial_m\Phi_s, \qquad \frac{1}{\mu}\alpha_1^2\alpha_3\sigma_s^{3m} = \left(k_s + \frac{\alpha_3^2}{\alpha_1^2}\right)\partial_3\partial_m\Phi_s$$

$$\frac{1}{\mu}\alpha_1^2\alpha_3\,\sigma_s^{33} = \left[(4-2q)\,k_s - k_s^{-1}\frac{\alpha_1^2}{\alpha_3^2} - (1-2q)\frac{\alpha_3^2}{\alpha_1^2}\right]\partial_3^2\Phi_s \;\; (s = 1,2;\; m = 1,2)$$

where the k_s are expressed in terms of the ζ_s^2 with the aid of (8.9).

For the solution (8.13), the expressions for the components of this tensor are

$$\frac{1}{\mu}\alpha_1^2\alpha_3\sigma^{11} = -\frac{1}{\mu}\alpha_1^2\alpha_3\sigma^{22} = 2\partial_1\partial_2\Psi$$

$$\frac{1}{\mu}\alpha_1^2\alpha_3\sigma^{12} = \frac{1}{\mu}\alpha_1^2\alpha_3\sigma^{21} = (\partial_2^2 - \partial_1^2)\,\Psi = (D^2 - 2\partial_1^2)\,\Psi = (2\partial_2^2 - D^2)\,\Psi$$

$$\frac{1}{\mu}\alpha_1^2\alpha_3\sigma^{13} = \partial_2\partial_3\Psi, \qquad \frac{1}{\mu}\alpha_1^2\alpha_3\sigma^{23} = -\partial_1\partial_3\Psi, \qquad \sigma^{33} = 0 \tag{8.16}$$

$$\frac{1}{\mu}\alpha_1^2\alpha_3\sigma^{31} = \frac{\alpha_3^2}{\alpha_1^2}\partial_2\partial_3\Psi, \qquad \frac{1}{\mu}\alpha_1^2\alpha_3\sigma^{32} = -\frac{\alpha_3^2}{\alpha_1^2}\partial_1\partial_3\Psi$$

The boundary conditions on the lateral surface of the bar are now written in the form

$$\sum_{s=1}^{2}\left[N_1\frac{1-2q}{1-q}\left(\frac{1}{\zeta_s^2}\frac{\alpha_3^2}{\alpha_1^2} - 1\right)D^2\Phi_s + 2\frac{\partial}{\partial N}\partial_1\Phi_s\right] - N_2D^2\Psi + 2\frac{\partial}{\partial N}\partial_2\Psi = 0$$

$$\sum_{s=1}^{2}\left[N_2\frac{1-2q}{1-q}\left(\frac{1}{\zeta_s^2}\frac{\alpha_3^2}{\alpha_1^2} - 1\right)D^2\Phi_s + 2\frac{\partial}{\partial N}\partial_2\Phi_s\right] + N_1D^2\Psi - 2\frac{\partial}{\partial N}\partial_1\Psi = 0 \tag{8.17}$$

$$\partial_3\left[\sum_{s=1}^{2}(k_s + 1)\frac{\partial\Phi_s}{\partial N} + \frac{\partial\Phi}{\partial\sigma}\right] = 0$$

$$\frac{\partial}{\partial N} = N_1\partial_1 + N_2\partial_2, \qquad \frac{\partial}{\partial \sigma} = N_1\partial_2 - N_2\partial_1$$

where the operational notation is introduced. These are proportional to the normal and tangential derivatives with respect to the arc of the contour of the cross section of the bar. Solutions of Equations (8.10) and (8.14) are sought in the form

$$\Phi_s = Z(a_3)\,\varphi_s(a_1, a_2), \qquad \Psi = Z(a_3)\,\psi(a_1, a_2) \tag{8.18}$$

and separation of the variable a_3 leads to the differential equations

$$Z'' + \lambda^2 Z = 0, \qquad D^2\varphi_s - \lambda^2\zeta_s^2\varphi_s = 0, \qquad D^2\Psi - \frac{\alpha_3^2}{\alpha_1^2}\lambda^2\psi = 0 \tag{8.19}$$

Cancelling the factor depending on the variable a_3, we now write the boundary conditions (8.17) for the functions φ_s and ψ in the form

$$\sum_{s=1}^{2}\left[\frac{1-2q}{2(1-q)}\left(\frac{\alpha_3^2}{\alpha_1^2} - \zeta_s^2\right)\varphi_s + \frac{1}{\lambda^2}\frac{\partial^2\varphi_s}{\partial N^2}\right] + \frac{1}{\lambda^2}\frac{\partial}{\partial N}\frac{\partial\psi}{\partial\sigma} = 0$$

$$\sum_{s=1}^{2}\frac{1}{\lambda^2}\frac{\partial}{\partial N}\frac{\partial\varphi_s}{\partial\sigma} + \frac{1}{2}\frac{\alpha_3^2}{\alpha_1^2}\psi - \frac{1}{\lambda^2}\frac{\partial^2\psi}{\partial N^2} = 0, \quad \sum_{s=1}^{2}(k_s+1)\frac{\partial\varphi_s}{\partial N} + \frac{\partial\psi}{\partial\sigma} = 0 \tag{8.20}$$

where

$$\frac{\partial^2}{\partial N^2} = N_1^2\partial_1^2 + 2N_1N_2\partial_1\partial_2 + N_2^2\partial_2^2, \qquad \frac{\partial}{\partial N}\frac{\partial}{\partial\sigma} = N_1N_2(\partial_2^2 - \partial_1^2) + (N_1^2 - N_2^2)\partial_1\partial_2$$

The assumption

$$Z = \cos\frac{\pi a_3}{2l} \qquad \left(\lambda = \frac{\pi}{2l}\right) \tag{8.21}$$

satisfies the third condition of (8.5) on the upper end; the two remaining conditions are satisfied on the average. This follows from the equilibrium equations (8.3) and the boundary conditions (8.4), since over the whole bar

$$\partial_3\iint \sigma^{3m}dO + \oint (N_1\sigma^{1m} + N_2\sigma^{2m})\,d\sigma = 0, \qquad \iint \sigma^{3m}\,dO = \text{const}$$

and this constant is equal to zero, since $\sigma^{3m} = 0$ for $a_3 = 0$, which is a consequence of (8.15), (8.16) (8.18) and (8.21). It is also easy to prove that no torsional moment is present; its derivative ∂_3 is also equal to zero for all a_3

$$\partial_3\iint (a_1\sigma^{32} - a_2\sigma^{31})\,dO = -\iint [a_1(\partial_1\sigma^{12} + \partial_2\sigma^{22}) - a_2(\partial_1\sigma^{11} + \partial_2\sigma^{21})]\,dO =$$
$$= \iint (\sigma^{12} - \sigma^{21})\,dO = 0$$

We note in addition that at the lower end $w_3 = 0$, $\partial_3 w_1 = \partial_3 w_2 = 0$ from (8.7) and (8.13); w_1 and w_2 can be made equal to zero at some point of the end $a_3 = 0$ by the choice of the constants w_1° and w_2°. In this sense the end $a_3 = 0$ is "clamped".

9. A bar of circular cross section. With the object in view of examining an equilibrium shape in which the axis of the bar does not remain straight, let us assume

$$\varphi_s = R_s(\lambda r)\cos\theta, \qquad \psi = R(\lambda r)\sin\theta \qquad \left(r = \sqrt{a_1^2 + a_2^2},\ \cos\theta = \frac{a_1}{r},\ \sin\theta = \frac{a_2}{r}\right) \tag{9.1}$$

The functions R_s and R are determined in accordance with (8.19) by Equations

$$R_s'' + \frac{1}{x}R_s' - \left(\zeta_s^2 + \frac{1}{x^2}\right)R_s = 0, \quad R'' + \frac{1}{x}R' - \left(\frac{\alpha_3^2}{\alpha_1^2} + \frac{1}{x^2}\right)R = 0 \quad (x = \lambda r) \tag{9.2}$$

The solutions of these equations that are finite at $x = 0$ are expressed in terms of a Bessel function with imaginary argument by

$$R_s = C_s I_1(\zeta_s x), \qquad R = C I_1\left(\frac{\alpha_3}{\alpha_1}x\right) \tag{9.3}$$

where the constants C_s are complex conjugates and C is real. Substitution .20), taking (8.9) into account, leads to a system of equations which is linear in these constants

$$\sum_{s=1}^{2} C_s \left(k_s + \frac{\alpha_3^2}{\alpha_1^2}\right) I_1(\rho_s) + \frac{\alpha_3^2}{\alpha_1^2} C I_1(\rho) = 0$$

$$\sum_{s=1}^{2} C_s \rho_s I_2(\rho_s) + \frac{1}{2} C \rho^2 I_1(\rho)\left[1 - \frac{2I_2(\rho)}{\rho I_1(\rho)}\right] = 0 \qquad \left(\rho_s = \zeta_s x_0, \ \rho = \frac{\alpha_3}{\alpha_1} x_0\right) \quad (9.4)$$

$$\sum_{s=1}^{2} C_s (k_s + 1)[\rho_s I_0(\rho_s) - I_1(\rho_s)] + C I_1(\rho) = 0$$

In obtaining this we have used the differential equations (9.2), well known transformations of Bessel functions, and the relation (8.9).

Equating the determinant of the system of equations (9.4) to zero, we arrive at an equation which relates the parameters x_0^2 and α_3^2/α_1^2. After dividing out $(\zeta_2^2 - \zeta_1^2)$, this equation reduces to real form, since ζ_1^2 and ζ_2^2 then enter only in terms of their sum and product. The computations are cumbersome, but only small values of the quantities x_0^2 and $1 - \alpha_3^2/\alpha_1^2 = \epsilon$ which are of the same order of magnitude are of interest. Then $\frac{1}{2}(\zeta_1^2 + \zeta_2^2)$ and $\zeta_1^2 \zeta_2^2$ differ from unity by terms of order ϵ, according to (8.11).

Retaining only linear and quadratic terms in ϵ and x_0^2, we have

$$2(1-q)\epsilon - {}^1/_2 x_0^2 (3 - 4q) - 2(1-q)\epsilon^2 + [{}^{23}/_6 (1-q) + {}^{25}/_{12}]\epsilon x_0^2 + \quad (9.5)$$

$$+ [{}^1/_{12} - {}^5/_6 (1-q)] x_0^4 = 0$$

Dropping nonlinear terms, we find in the first approximation that

$$1 - \frac{\alpha_3^2}{\alpha_1^2} = \epsilon = \frac{x_0^2}{2} \frac{3-4q}{2(1-q)} = \frac{x_0^2}{2}(1+\nu) \quad (9.6)$$

Thus $x_0^2 > 0$ for $\epsilon > 0$, i.e. for a compressed bar. Returning to Equations (4.10) and (4.11) we find in this approximation

$$1 - \alpha_3^2 \approx {}^1/_2 x_0^2, \quad |Q| = ES_0 \frac{x_0^2}{4} = ES_0 \frac{r_0^2 \lambda^2}{4} = EJ\lambda^2 = Q_9 \quad (9.7)$$

where, as was to be expected, Q_9 is the Euler critical value of the compressive force (J is the moment of inertia of the cross section).

In the second approximation

$$\epsilon = {}^1/_2 x_0^2 (1+\nu) - {}^{17}/_{12} (1 + {}^9/_{34}\nu - {}^{31}/_{34}\nu^2) x_0^4 \quad (9.8)$$

and again turning to Equations (4.10) and (4.11), we find

$$\frac{|Q|}{Q_9} = 1 - \frac{1}{2} x_0^2 \left(\frac{1}{2} - \nu + \frac{17}{3} \frac{1 + {}^9/_{34}\nu - {}^{31}/_{34}\nu^2}{1+\nu}\right) \quad (9.9)$$

so that for $\nu = 0$ and $\nu = \frac{1}{2}$ we have, respectively,

$$\frac{|Q|}{Q_9} \approx 1 - 3.08 x_0^2, \qquad \frac{|Q|}{Q_9} \approx 1 - 2.56 x_0^2 \quad (9.10)$$

10. Comments and references to the literature. The general tensorial relations of the nonlinear theory of elasticity which were briefly enumerated in Sections 1 to 4 are explained in the books [1 to 3] and in paper [4]. The latter reference also contains an exhaustive bibliography on nonlinear continuum mechanics up to 1953.

A derivation of the differential equations of equilibrium shapes near a given equilibrium state using the energy method is given in [5] and in Chapter IX of [6]. A direct construction of the equations of statics of an initially stressed medium in the case of a general state of stress is carried out in Chapter IV of [1], and one based on an intuitive geometric method is given in [7]. The latter is presented in greater detail in the book [8]. The energy approach to the problem is also developed in the well known papers [9 and 10]. The author of this paper is not aware of any works in which the bifurcation values of load parameters are found on the basis of an examination of a three-dimensional problem of equilibrium of an initially stressed elastic body (Sections 8 and 9 of the present paper).

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